Suffolk County Community College

Michael J. Grant Campus

Department of Mathematics

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MAT 101
A Survey of Mathematical Reasoning

Final Exam: Solutions and Answers

Instructor:

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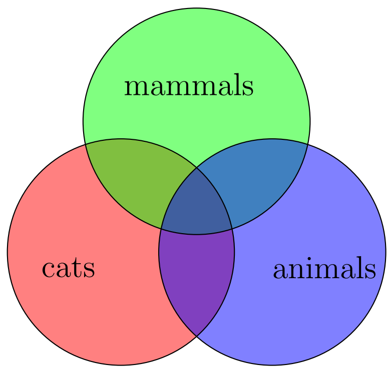
Existential presupposition is in effect for *(some x are y)* statements. The proposition *(some cats are smart)* means that the universe of discourse contains at least one cat which is smart. Existential presupposition is not in effect for *(all x are y)* statements. The proposition *(all unicorns are mammals)* means that every unicorn existing in the universe of discourse must be a mammal, but it does not assume that even a single unicorn exists.

Problem 1. Consider the following syllogism:

Some cats are mammals.
Some mammals are animals.
———
Some cats are animals.

(1). Draw a Venn diagram showing the categories of this syllogism.

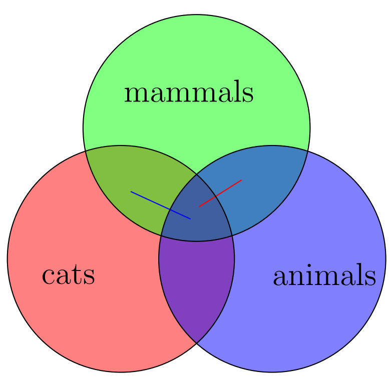
Space for your solution:



(2). Express the assumptions of this syllogism graphically in the Venn diagram.

Space for your solution:

In the diagram below, the blue interval represents the assumption *(some cats are mammals)* and the red one — *(some mammals are animals)*.



(3). Give a reason why this syllogism is a valid argument, or provide a counterexample showing that it is invalid.

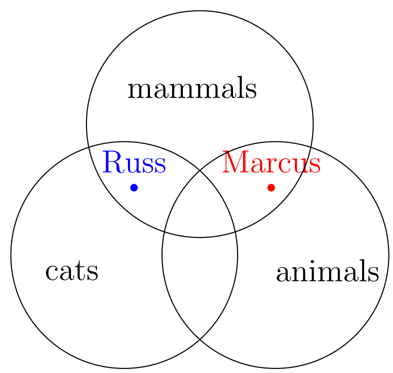
Space for your solution:

Consider a universe of discourse consisting of two objects, Marcus and Russ, having the following properties:

- Marcus is a mammalian animal who is not a cat; and
- Russ is a mammalian cat who is not an animal.

(We can model this situation in our own world by taking two people: father Marcus and son Russ — as the universe of discourse, and by assuming that "mammal" means "male", "cat" means "child" and "animal" means "ancestor". Then Marcus is a male ancestor who is not a child of anybody in the universe, and Russ is a male child who is not an ancestor of anybody in the universe.)

This universe satisfies the assumptions of the argument, but violates its conclusion, providing a *counterexample* for it. Thus, the conclusion does not follow from the assumptions, and the argument in question is invalid.



(4). Using predicates and quantifiers, express this syllogism in the form of a Gentzen-style argument of predicate logic.

Space for your solution:

$$\frac{\begin{array}{l}\exists x : \text{cat}(x) \bigwedge \text{mammal}(x)\\ \exists x : \text{mammal}(x) \bigwedge \text{animal}(x)\end{array}}{\exists x : \text{cat}(x) \bigwedge \text{animal}(x)}$$

Problem 2. Suppose three suspects were caught in an art museum. Before they surrendered to police, they agreed that each one of them will tell a half-truth to their interrogators. They were separated and asked two questions: which painting they wanted to steal and who commissioned them.

- The first said that they wanted to steal the Rembrandt and the crime boss Big Joe promised to buy it.
- The second said that they came to get the Degas and stated that it was definitely not the Big Joe who sent them.
- The last suspect claimed that they came to steel the Monet, and were commissioned by the notorious Lucky Sam.

(1). Using disjunction to represent a half-true statement, we can write the information conveyed by the first suspect as

$$\text{Rembrandt} \vee \text{Joe}.$$

Assuming, in addition to the information gained in the interrogations, that the Rembrandt was not the target of the theft, formulate a valid argument in Gentzen's notation:

$$\begin{array}{l} \text{Rembrandt} \vee \text{Joe} \\ \neg\text{Rembrandt} \\ \hline ??? \end{array}$$

stating who commissioned the theft. For this problem, only the statement of the argument, rather than verification of its validity, is needed.

Space for your solution:

$$\begin{array}{l} \text{Rembrandt} \vee \text{Joe} \\ \neg\text{Rembrandt} \\ \hline \text{Joe} \end{array}$$

(2). Write the argument which you gave as the answer to the previous problem as a single statement of propositional logic. (Use any necessary logical connectors that may be implicit in Gentzen's notation.)

Space for your solution:

$$\Big(\big(\text{Rembrandt} \vee \text{Joe}\big) \wedge \big(\neg\text{Rembrandt}\big)\Big) \Rightarrow \text{Joe}.$$

(3). Use the truth tables to verify the validity of this argument.

Space for your solution:

The truth table (with "R" denoting "Rembrandt") is:

R	Joe	$\Big(\big(\text{Rembrandt} \vee \text{Joe}\big) \wedge \big(\neg\text{Rembrandt}\big)\Big) \Rightarrow \text{Joe}$
T	T	$\big[\big((T \vee T) \wedge \neg T\big) \Rightarrow T\big] = \big[\big(T \wedge \neg T\big) \Rightarrow T\big] = \big[\big(T \wedge F\big) \Rightarrow T\big] = [F \Rightarrow T] = T$
T	F	$\big[\big((T \vee F) \wedge \neg T\big) \Rightarrow F\big] = \big[\big(T \wedge \neg T\big) \Rightarrow F\big] = \big[\big(T \wedge F\big) \Rightarrow F\big] = [F \Rightarrow F] = T$
F	T	$\big[\big((F \vee T) \wedge \neg F\big) \Rightarrow T\big] = \big[\big(T \wedge \neg F\big) \Rightarrow T\big] = \big[\big(T \wedge T\big) \Rightarrow T\big] = [T \Rightarrow T] = T$
F	F	$\big[\big((F \vee F) \wedge \neg F\big) \Rightarrow F\big] = \big[\big(F \wedge \neg F\big) \Rightarrow F\big] = \big[\big(F \wedge T\big) \Rightarrow F\big] = [F \Rightarrow F] = T$

Since the last column contains only the truths, the argument is a tautology and thus valid.

(4). Suppose we record the information gained in the interrogation in the form of the following Gentzen-style argument:

$$\begin{array}{l} \text{Rembrandt} \vee \text{Joe} \\ \text{Degas} \vee (\neg\text{Joe }) \\ \text{Monet} \vee \text{Sam} \\ \hline \text{???} \end{array}$$

Assume that in addition to this information the police knows that the thiefs could only steal one painting, and that Big Joe and Lucky Sam are enemies, so the the thiefs could only serve one of them but not both. How can that additional information be added to the above assumptions?

Space for your solution:

The fact that the crime bosses and targeted artists are mutually exclusive can be stated by adding the following four negation statements at the end of the assumptions list:

$$\begin{array}{l} \text{Rembrandt} \vee \text{Joe} \\ \text{Degas} \vee (\neg\text{Joe }) \\ \text{Monet} \vee \text{Sam} \\ \neg(\text{Monet} \wedge \text{Rembrandt}) \\ \neg(\text{Rembrandt} \wedge \text{Degas}) \\ \neg(\text{Degas} \wedge \text{Monet}) \\ \neg(\text{Joe} \wedge \text{Sam}) \\ \hline \text{???} \end{array}$$

(5). EXTRA CREDIT Use the distributivity of conjunction with respect to disjunction:

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C),$$

to extract the truth from the information depicted in the previous subproblem. Hint: you may find it convenient two write conjunction as multiplication, and disjunction as addition, so that the above information gets recorded as

$$\begin{array}{l} \text{Rembrandt} + \text{Joe} \\ \text{Degas} + (\neg\text{Joe}\ ) \\ \text{Monet} + \text{Sam} \\ \ldots \\ \hline ??? \end{array}$$

It may also be helpful to write the falsehood of a statement X as $X = 0$, and its truth as $X = 1$.

Space for your solution:

$$1 = \Big(\text{Rembrandt} + \text{Joe}\Big) \cdot \Big(\text{Degas} + (\neg\text{Joe})\Big) \cdot \Big(\text{Monet} + \text{Sam}\Big) =$$

$$= \boxed{\text{open parentheses using the distributivity}} =$$

$$\text{Rembrandt} \cdot \text{Degas} \cdot \text{Monet} + \text{Rembrandt} \cdot \text{Degas} \cdot \text{Sam} +$$
$$\text{Rembrandt} \cdot (\neg\text{Joe}) \cdot \text{Monet} + \text{Rembrandt} \cdot (\neg\text{Joe}) \cdot \text{Sam} +$$
$$\text{Joe} \cdot \text{Degas} \cdot \text{Monet} + \text{Joe} \cdot \text{Degas} \cdot \text{Sam} +$$
$$\text{Joe} \cdot (\neg\text{Joe}) \cdot \text{Monet} + \text{Joe} \cdot (\neg\text{Joe}) \cdot \text{Sam} =$$

$$= \boxed{\text{two or more painters or crime bosses in one term make it zero}} =$$

$$0 + 0 + 0 + \text{Rembrandt} \cdot (\neg\text{Joe}) \cdot \text{Sam} + 0 + 0 + 0 + 0.$$

Thus, going back to the usual logic notation, we can conclude that

$$\begin{array}{l} \text{Rembrandt} \vee \text{Joe} \\ \text{Degas} \vee (\neg\text{Joe}\ ) \\ \text{Monet} \vee \text{Sam} \\ \neg(\text{Monet} \wedge \text{Rembrandt}) \\ \neg(\text{Rembrandt} \wedge \text{Degas}) \\ \neg(\text{Degas} \wedge \text{Monet}) \\ \neg(\text{Joe} \wedge \text{Sam}) \\ \hline \text{Rembrandt} \wedge (\neg\text{Joe}) \wedge \text{Sam} \end{array}$$

is a valid propositional argument. In other words, the thiefs wanted to steal the Rembrand and were working for the Lucky Sam.

Problem 3. Consider the following argument: *Iron is gradually sinking from the mantle of earth into its core. Since iron is heavier than the rocks it displaces, this sinking decreases earth's moment of inertia. Decreasing the moment of inertia of any closed system increases its speed of rotation* [1]. *If the speed of earth's rotation were to increase, days would get shorter. In fact, days are getting longer, not shorter. Thus, the earth is not a closed system.*

(1). Identify and properly name [2] any single gate used (explicitly or implicitly) in this argument.

Space for your solution:

Any single item from the following list qualifies as a full answer:

- "Thus" in the last sentence is an explicit *implication*;
- every full stop at the end of a sentence (other than the one one before "Thus") is an implicit *conjunction*;
- "Decreasing... increases..." passage has an implicit *implication* since it can be rephrased as: "if you decrease..., then... will increase";
- "If the speed..." is an explicit *implication*;
- the bounded universal quantifier "any closed system" will include an *implication* when translated into the language of predicate logic.

(2). Identify any single predicate used in this argument.

Space for your solution:

Any single item from the following list qualifies as a full answer:

- $c(x) = (x \text{ is a closed system})$,
- $i(x) = (\text{the moment of inertia of } x \text{ is decreasing})$,
- $r(x) = (\text{rotational speed of } x \text{ is increasing})$.

[1] This follows from the law of physics, called conservation of angular momentum. The law states that the angular momentum of a closed system remains constant over time. Since the angular momentum is the product of the moment of inertia and of the rotational speed, decrease in one of those two parameters (of a given closed system) will result in the proportional increase in another.

[2] i.e. as a "negation", "conjunction", "disjunction", "implication", or "equivalence"

(3). Translate this argument into the format of predicate logic [3].

Space for your solution:

Denote:

- the statement $S =$ (iron is sinking from the mantle of earth into its core),
- the statement $D =$ (days are getting shorter),
- the object $E =$ (the earth),
- the predicate $c(x) =$ (x is a closed system),
- the predicate $i(x) =$ (the moment of inertia of x is decreasing),
- the predicate $r(x) =$ (rotational speed of x is increasing).

Using these notations, the original argument can be written as:

$$\begin{array}{l} S \\ S \Rightarrow i(E) \\ \forall x : c(x) \Rightarrow \big(i(x) \Rightarrow r(x)\big) \\ r(E) \Rightarrow D \\ \neg D \\ \hline \neg c(E) \end{array}$$

[3] in Gentzen's notation

(4). Prove the validity of this argument.

Space for your solution:

1. S — assumption;
2. $S \Rightarrow i(E)$ — assumption;
3. $\forall x : c(x) \Rightarrow \Big(i(x) \Rightarrow r(x)\Big)$ — assumption;
4. $r(E) \Rightarrow D$ — assumption;
5. $\neg D$ — assumption;
6. $c(E) \Rightarrow \Big(i(E) \Rightarrow r(E)\Big)$ — deduced from 3 via *universal instantiation* [a]
$$\frac{\forall x : P(x); \quad c : \text{object}}{P(c)}$$
7. $i(E)$ — deduced from 2 and 1 via *modus ponens* [b]
$$\frac{\begin{array}{l} A \Rightarrow B \\ A \end{array}}{B}$$
8. $\neg r(E)$ — deduced from 4 and 5 via via *modus tollens*
$$\frac{\begin{array}{l} A \Rightarrow B \\ \neg B \end{array}}{\neg A}$$
9. (hyp) $c(E)$ — undischarged hypothesis starting a sub-proof;
10. (sub) $i(E) \Rightarrow r(E)$ — deduced from 6 and 9 via *modus ponens*;
11. (sub) $r(E)$ — deduced from 10 and 7 via *modus ponens*;
12. (sub) F — deduced from 11 and 8 via *the law of contradiction* [c]
$$\frac{\begin{array}{l} A \\ \neg A \end{array}}{F}$$
13. $c(E) \Rightarrow F$ — deduced from 9 and 12 via the *hypothesis discharge* [d]
14. $\neg c(E)$ — deduced from 13 via *proof by contradiction* [e] (this can be seen as a particular case of modus tollens)
$$\frac{A \Rightarrow F}{\neg A}$$

[a] or *elimination of universal* in Gentzen's terminology
[b] or *elimination of implication* in Gentzen's terminology
[c] or *elimination of negation* in Gentzen's terminology
[d] or *introduction of implication* in Gentzen's terminology
[e] or *introduction of negation* in Gentzen's terminology