Suffolk County Community College
Michael J. Grant Campus

Department of Mathematics

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MAT 111
Algebra-II

Final Exam: Solutions and Answers

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Problem 1. In this problem we will, through a series of sub-problems, solve the system of equations

$$\begin{cases} 2x + 7y + 2z = -9 \\ 3x + 11y - 2z = 8 \\ 8x + 29y - 3z = 11 \end{cases}$$

using Gauß-Jordan method. Any other solution will not be accepted, so please keep your work relevant to the specific question being asked in each sub-problem.

(1). Write the augmented matrix of this system of equations.

Space for your solution:

$$\left[\begin{array}{rrr|r} 2 & 7 & 2 & -9 \\ 3 & 11 & -2 & 8 \\ 8 & 29 & -3 & 11 \end{array}\right]$$

(2). Find elementary row transformations that would make the top left corner of the augmented matrix equal to 1, while avoiding any fractions in the resulting matrix. Perform these transformations and make them explicit by using the R_i notation.

Space for your solution:

This task can be accomplished by subtracting the first row from the second row, followed by switching the first and the second rows:

$$\left[\begin{array}{rrr|r} 2 & 7 & 2 & -9 \\ 3 & 11 & -2 & 8 \\ 8 & 29 & -3 & 11 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \sim$$

$$\begin{array}{l} R_1 \\ R_2 - R_1 \\ R_3 \end{array} \left[\begin{array}{rrr|r} 2 & 7 & 2 & -9 \\ 1 & 4 & -4 & 17 \\ 8 & 29 & -3 & 11 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \sim$$

$$\begin{array}{l} R_2 \\ R_1 \\ R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 4 & -4 & 17 \\ 2 & 7 & 2 & -9 \\ 8 & 29 & -3 & 11 \end{array}\right]$$

(3). Add multiples of the first row to the second and third row (as needed) to make the first column entries of these two rows equal to zero. Make your row transformations explicit by using the R_i notation.

Space for your solution:

$$\left[\begin{array}{ccc|c} 1 & 4 & -4 & 17 \\ 2 & 7 & 2 & -9 \\ 8 & 29 & -3 & 11 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \sim \begin{array}{l} R_1 \\ R_2 - 2 \cdot R_1 \\ R_3 - 8 \cdot R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 4 & -4 & 17 \\ 0 & -1 & 10 & -43 \\ 0 & -3 & 29 & -125 \end{array}\right]$$

(4). Use elementary row transformations to turn the matrix into the so-called *row echelon form* by making all entries on the diagonal equal to 1 and all entries below the diagonal equal to 0. Make the row transformations explicit by using the R_i notation.

Space for your solution:

$$\left[\begin{array}{ccc|c} 1 & 4 & -4 & 17 \\ 0 & -1 & 10 & -43 \\ 0 & -3 & 29 & -125 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \sim$$

$$\begin{array}{l} R_1 \\ -R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 4 & -4 & 17 \\ 0 & 1 & -10 & 43 \\ 0 & -3 & 29 & -125 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \sim$$

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 + 3 \cdot R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 4 & -4 & 17 \\ 0 & 1 & -10 & 43 \\ 0 & 0 & -1 & 4 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \sim$$

$$\begin{array}{l} R_1 \\ R_2 \\ -R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 4 & -4 & 17 \\ 0 & 1 & -10 & 43 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

(5). Use elementary row transformations to turn into zero all entries above the leader of the third row. Make the row transformations explicit by using the R_i notation.

Space for your solution:

$$\left[\begin{array}{ccc|c} 1 & 4 & -4 & 17 \\ 0 & 1 & -10 & 43 \\ 0 & 0 & 1 & -4 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \sim \begin{array}{l} R_1 + 4 \cdot R_3 \\ R_2 + 10 \cdot R_3 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 4 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

(6). Use elementary row transformations to turn into zero all entries above the leader of the second row. Make the row transformations explicit by using the R_i notation.

Space for your solution:

$$\left[\begin{array}{ccc|c} 1 & 4 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \sim \begin{array}{l} R_1 - 4 \cdot R_2 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

(7). Based on the answer to the previous sub-problem, determine the solution of the original system:

$$\begin{cases} x = \\ y = \\ z = \end{cases}$$

Space for your solution:

$$\begin{cases} x = -11 \\ y = 3 \\ z = -4 \end{cases}$$

Problem 2. Solve the inequality:

$$2x + |3x - 5| \geq 10.$$

Space for your solution:

$$2x + |3x-5| \geq 10 \quad \Leftrightarrow \quad \left[\begin{array}{l} \left\{ \begin{array}{l} 3x-5 \geq 0 \\ 2x + |3x-5| \geq 10 \end{array} \right. \\ \\ \left\{ \begin{array}{l} 3x-5 < 0 \\ 2x - |3x-5| \geq 10 \end{array} \right. \end{array} \right. \quad \Leftrightarrow \quad \left[\begin{array}{l} \left\{ \begin{array}{l} 3x-5 \geq 0 \\ 2x + (3x-5) \geq 10 \end{array} \right. \\ \\ \left\{ \begin{array}{l} 3x-5 < 0 \\ 2x - (3x-5) \geq 10 \end{array} \right. \end{array} \right. \quad \Leftrightarrow$$

$$\left[\begin{array}{l} \left\{ \begin{array}{l} 3x \geq 5 \\ 2x + 3x - 5 \geq 10 \end{array} \right. \\ \\ \left\{ \begin{array}{l} 3x < 5 \\ 2x - 3x + 5 \geq 10 \end{array} \right. \end{array} \right. \quad \Leftrightarrow \quad \left[\begin{array}{l} \left\{ \begin{array}{l} 3x \geq 5 \\ 5x \geq 15 \end{array} \right. \\ \\ \left\{ \begin{array}{l} 3x < 5 \\ -x \geq 5 \end{array} \right. \end{array} \right. \quad \Leftrightarrow \quad \left[\begin{array}{l} \left\{ \begin{array}{l} x \geq \frac{5}{3} \\ x \geq 3 \end{array} \right. \\ \\ \left\{ \begin{array}{l} x < \frac{5}{3} \\ x \leq -5 \end{array} \right. \end{array} \right. \quad \Leftrightarrow \quad \left[\begin{array}{l} x \geq 3 \\ x \leq -5 \end{array} \right. \quad \Leftrightarrow$$

$$x \in (-\infty, -5] \cup [3, +\infty).$$

Problem 3. Consider the quadratic polynomial $-3x^2 + 6x + 15$.

(1). Perform the completion of the square.

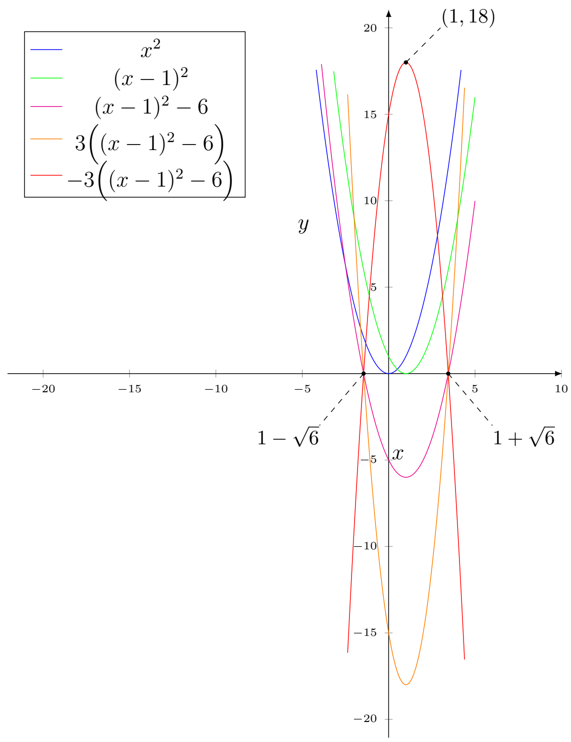
Space for your solution:

$$-3x^2+6x+15 = -3\Big(x^2-2x-5\Big) = -3\Big(x^2+2x(-1)+(-1)^2-(-1)^2-5\Big) = -3\Big((x-1)^2-6\Big).$$

(2). Sketch the graph of the function $y = f(x)$ on the (x, y)-coordinate plane by transforming the graph of $y = x^2$.

Space for your solution:

(The vertex and the x-intercepts in the following picture will be found in subsequent sub-problems.)



(3). Use the result of completion of the square to find the vertex of the parabola, and mark it on the sketch in sub-problem 2.

Space for your solution:

1. The original parabola $y = x^2$ has vertex $(0, 0)$.
2. Parabola $(x-1)^2$ is the result of shifting $y = x^2$ right by 1, so its vertex moves from $(0, 0)$ to $(1, 0)$.
3. Parabola $(x-1)^2 - 6$ is the result of shifting $(x-1)^2$ down by 6, so its vertex moves from $(1, 0)$ to $(1, -6)$.
4. Parabola $3\Big((x-1)^2 - 6\Big)$ is the result of streching $(x-1)^2 - 6$ by factor of 3 away from x-axis, so its vertex moves from $(1, -6)$ to $(1, -18)$.
5. Parabola $-3\Big((x-1)^2 - 6\Big)$ is the result of reflecting $3\Big((x-1)^2 - 6\Big)$ with respect to the x-axis, so its vertex moves from $(1, -18)$ to $(1, 18)$.

(4). Use the result of completion of the square and the difference-of-two-squares formula to find the roots of that polynomial, and mark them on the sketch in sub-problem 2.

Space for your solution:

Continuing from the completion of the square:

$$-3\Big((x-1)^2 - 6\Big) = -3\Big((x-1)^2 - \left(\sqrt{6}\right)^2\Big) = -3\left(x - 1 - \sqrt{6}\right)\left(x - 1 + \sqrt{6}\right).$$

Therefore:
$-3x^2 + 6x + 15 = 0 \quad \Leftrightarrow$

$$-3\left(x - 1 - \sqrt{6}\right)\left(x - 1 + \sqrt{6}\right) = 0 \quad \Leftrightarrow \quad \left[\begin{array}{l} x - 1 - \sqrt{6} = 0 \\ x - 1 + \sqrt{6} = 0 \end{array}\right. \quad \Leftrightarrow \quad \left[\begin{array}{l} x = 1 + \sqrt{6} \\ x = 1 - \sqrt{6} \end{array}\right.$$