Suffolk County Community College
Michael J. Grant Campus

Department of Mathematics

Thursday, December 5, 2024 (returned Tuesday, December 10, 2024)

MAT 129
College Precalculus

Final Exam: Solutions and Answers

Instructor:
Name: Alexander Kasiukov
Office: Suffolk Federal Credit Union Arena, Room A-109
Phone: (631) 851-6484
Email: kasiuka@sunysuffolk.edu
Web Site: http://kasiukov.com

Problem 1. Suppose
set $A = \{\text{Paris}, \text{Ottawa}, \text{Toronto}, \text{Berlin}, \text{Madrid}\}$ and
set $B = \{\text{Canada}, \text{France}, \text{Germany}, \text{Spain}\}$. Define a function "`Country`" to have domain A, range B and graph

$$\{(\text{Paris}, \text{France}), (\text{Ottawa}, \text{Canada}), (\text{Toronto}, \text{Canada}), (\text{Berlin}, \text{Germany}), (\text{Madrid}, \text{Spain})\}.$$

(1). What is `Country`(Berlin)?

Space for your solution:

By definition, `Country`(Berlin) is the second element of the only ordered pair in the graph of the function `Country` that has "Berlin" as its first element. In our case, such a pair is the pair (Berlin, Germany), therefore `Country`(Berlin) = Germany.

(2). What is the image of the function `Country`?

Space for your solution:

By definition, the image of the function `Country` is the set of all the elements in its range that have the form `Country`(x), where x is an element of the domain of `Country`:

$$\begin{multline} \texttt{Im Country} = \\ \{\texttt{Country}(\text{Paris}), \texttt{Country}(\text{Ottawa}), \texttt{Country}(\text{Toronto}), \texttt{Country}(\text{Berlin}), \texttt{Country}(\text{Madrid})\} \\ = \{\text{France}, \text{Canada}, \text{Canada}, \text{Germany}, \text{Spain}\} \\ = \{\text{France}, \text{Canada}, \text{Germany}, \text{Spain}\}. \end{multline}$$

(3). Can the function `Country` be inverted? If yes, find the domain, range and graph of the inverse. If no, explain why.

Space for your solution:

The function `Country` cannot be inverted, because it is not one-to-one:

$$\texttt{Country}(\text{Ottawa}) = \text{Canada} = \texttt{Country}(\text{Toronto}).$$

In other words, the inverse relation violates the uniqueness condition of the vertical line test: the input "Canada" would have to result in two different outputs, "Ottawa" and "Toronto".

Problem 2. Consider the function with the range $\mathbb{R}$, defined by the formula

$$f(x) = \frac{2x^3 - 5x^2 + 7}{x^2 - 4x + 4}$$

for all $x \in \mathbb{R}$, for which the above formula makes sense.

(1). What is the domain of the function f?

Space for your solution:

The above formula makes sense if and only if the denominator $x^2 - 4x + 4$ of the fraction defining $f(x)$ is not zero. Since $x^2 - 4x + 4 = 0 \quad \Leftrightarrow \quad (x-2)^2 = 0 \quad \Leftrightarrow \quad x = 2$, we get:

$$\texttt{Dom}\ f = \{x \in \mathbb{R} : \ x \neq 2\}\,.$$

(2). Find all the vertical asymptotes of the graph of $f(x)$.

Space for your solution:

The graph of $f(x)$ has vertical asymptotes when the denominator of the fraction turns into zero (while the numerator stays non-zero). The denominator $x^2 - 4x + 4 = (x-2)^2$ has a single root $x = 2$ of multiplicity 2. Thus the line $x = 2$ is the only vertical asymptote of $f(x)$.

(3). Find the y-intercept of $f(x)$.

Space for your solution:

The y-intercept is the value of the function that corresponds to the input $x = 0$, thus the y-intercept is $f(0) = \frac{7}{4}$.

(4). Perform long division of the numerator of $f(x)$ by its denominator. Using the results of the long division, write $f(x)$ as a sum of a polynomial and a proper fraction.

Space for your solution:

$$\begin{array}{r|l} & \quad\quad\quad 2x+3 \\ x^2-4x+4 & \; 2x^3-5x^2+0x+7 \\ & -\left(2x^3-8x^2+8x\right) \\ \hline & \quad\quad 3x^2-8x+7 \\ & \quad -\left(3x^2-12x+12\right) \\ \hline & \quad\quad\quad 4x-5 \end{array}$$

The above long division means that

$$f(x) = 2x + 3 + \frac{4x-5}{x^2-4x+4}.$$

(5). Find the equation of the oblique asymptote of $f(x)$.

Space for your solution:

The oblique asymptote of $f(x)$ is the line $y = 2x + 3$.

(6). Find all the intersections of the graph of $f(x)$ with the oblique asymptote. (Only the x-coordinates of the intersections are needed.)

Space for your solution:

The graph of $f(x)$ will intersect the oblique asymptote whenever the numerator of $\frac{4x-5}{x^2-4x+4}$ turns into zero. The numerator has a single root $x = \frac{5}{4}$ of multiplicity 1.

(7). Use the Rational Roots Theorem to find a rational root of $2x^3 - 5x^2 + 7$.

Space for your solution:

The Rational Roots Theorem states that any rational root of this polynomial will be in the set:

$$\left\{\frac{\text{divisor of } 7}{\text{divisor of } 2}\right\} = \left\{1, -1, 7, -7, \frac{1}{2}, -\frac{1}{2}, \frac{7}{2}, -\frac{7}{2}\right\}.$$

Trying those values of x in $2x^3 - 5x^2 + 7$ yields the root $x = -1$.

(8). Use the result of the previous subproblem to find all x-intercepts of the funciton $f(x)$.

Space for your solution:

The x-intercepts of the function $f(x)$ are the roots of its numerator $2x^3 - 5x^2 + 7$. In the previous subproblem, we found one of its roots $x = -1$. The Polynomial Remainder Theorem of Bézout guarantees divisibility of $2x^3 - 5x^2 + 7$ by $x + 1$:

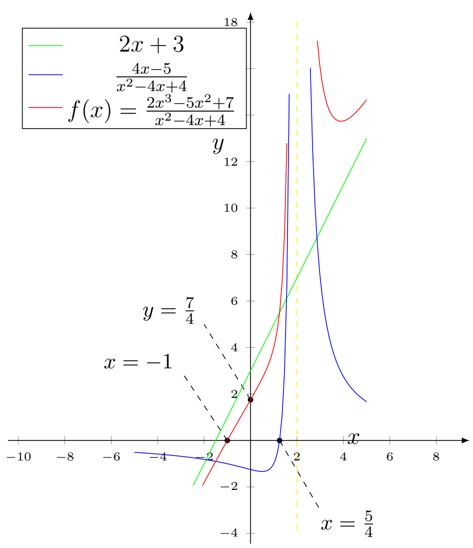
$$\begin{array}{r|l} & 2x^2 - 7x + 7 \\ \hline x+1 & 2x^3 - 5x^2 + 0x + 7 \\ - & 2x^3 + 2x^2 \\ \hline & -7x^2 + 0x \\ - & -7x^2 - 7x \\ \hline & 7x + 7 \\ - & 7x + 7 \\ \hline & 0 \end{array}$$

Thus $2x^3 - 5x^2 + 7 = 0 \quad \Leftrightarrow \quad (x+1)(2x^2 - 7x + 7) = 0 \quad \Leftrightarrow \quad \left[\begin{array}{l} x + 1 = 0 \\ 2x^2 - 7x + 7 = 0 \end{array}\right.$

The quadratic equation in the above system has no solution because its discriminant $D = b^2 - 4ac = (-7)^2 - 4 \cdot 2 \cdot 7 = 49 - 56 = -7$ is negative. Therefore $x = -1$ is the only x-intercept of the function $f(x)$.

(9). Use the result of the previous sub-problems to sketch the graph of the function f. Mark all vertical, horizontal and oblique asymptotes, as well as all intersections of the graph with the asymptotes and the axis, if any.

Space for your solution:



Problem 3. In this problem, we will consider functions $(\log_7 x) - 1$ and $\log_7(x+1)$.

(1). Solve the equation $(\log_7 x) - 1 = \log_7(x+1)$.

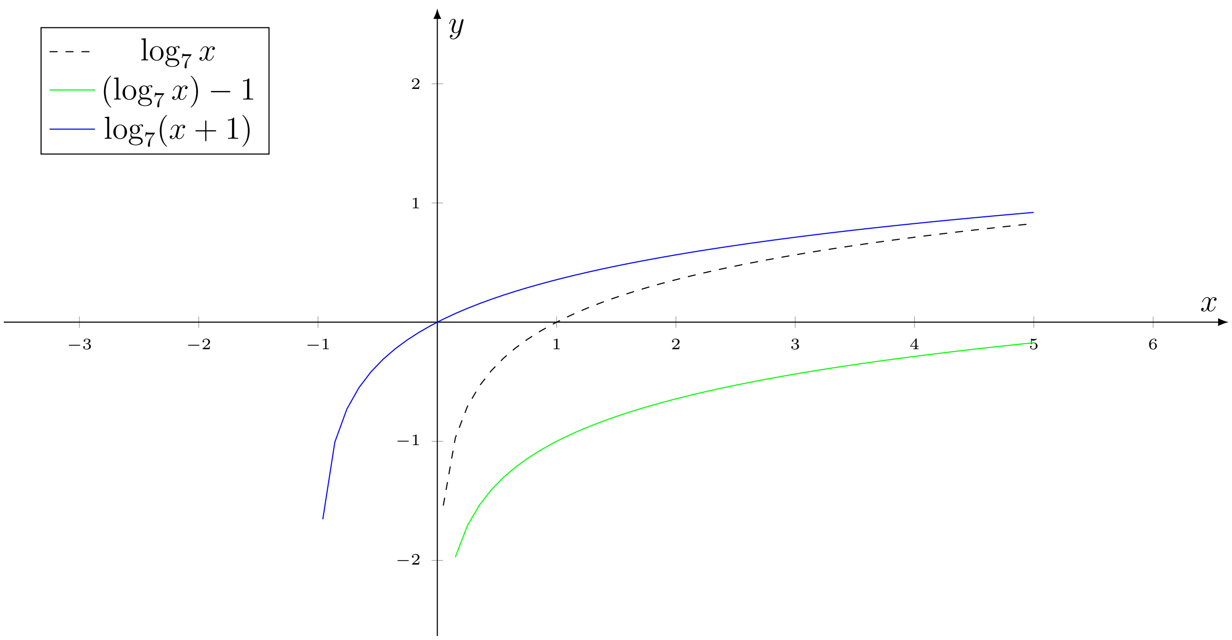
Space for your solution:

$$(\log_7 x) - 1 = \log_7(x+1) \Leftrightarrow 7^{(\log_7 x)-1} = 7^{\log_7(x+1)} \Leftrightarrow \frac{7^{(\log_7 x)}}{7^1} = 7^{\log_7(x+1)} \Leftrightarrow$$

$$\begin{cases} \frac{x}{7} = x+1 \\ x > 0 \\ x+1 > 0 \end{cases} \Leftrightarrow \begin{cases} x = 7x+7 \\ x > 0 \end{cases} \Leftrightarrow \begin{cases} -6x = 7 \\ x > 0 \end{cases} \Leftrightarrow \begin{cases} x = -\frac{7}{6} \\ x > 0 \end{cases} \Leftrightarrow x \in \emptyset.$$

(2). By transforming the graph of $\log_7 x$, sketch the graphs of these functions in the same (x, y)-coordinate system. Is this sketch is consistent with your solution of part (1)?

Space for your solution:



The graph of $(\log_7 x) - 1$ is the result of shifting down $\log_7 x$ by 1, whereas the graph of $\log_7(x+1)$ is obtained by shifting $\log_7 x$ left by 1. Since the graphs of $(\log_7 x) - 1$ and $\log_7(x+1)$ don't intersect, the equation

$$(\log_7 x) - 1 = \log_7(x+1)$$

has no solution, as already determined analytically in part (1).

Problem 4. Solve the equation $\cos(t) + \sin(t) = 0$.

Space for your solution:

$$\cos(t) + \sin(t) = 0 \quad \Leftrightarrow \quad \sin(t) = -\cos(t)$$

$$\Leftarrow\!\boxed{\text{divide both sides by } \cos(t)\text{, but remember about the possibility of it being zero}}\!\Rightarrow$$

$$\left[\begin{array}{l} \begin{cases} \cos(t) = 0 \\ \sin(t) = 0 \end{cases} \\ \\ \tan(t) = -1 \end{array}\right. \quad \Leftarrow\!\boxed{\text{the subsystem is incompatible}}\!\Rightarrow \quad \tan(t) = -1 \quad \Leftrightarrow$$

$$\exists n \in \mathbb{Z} : t = \arctan(-1) + \pi n \quad \Leftrightarrow \quad \exists n \in \mathbb{Z} : t = -\tfrac{\pi}{4} + \pi n.$$