Suffolk County Community College

Michael J. Grant Campus

Department of Mathematics

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MAT 141
Calculus with Analytic Geometry I

Final Exam: Solutions and Answers

Instructor:

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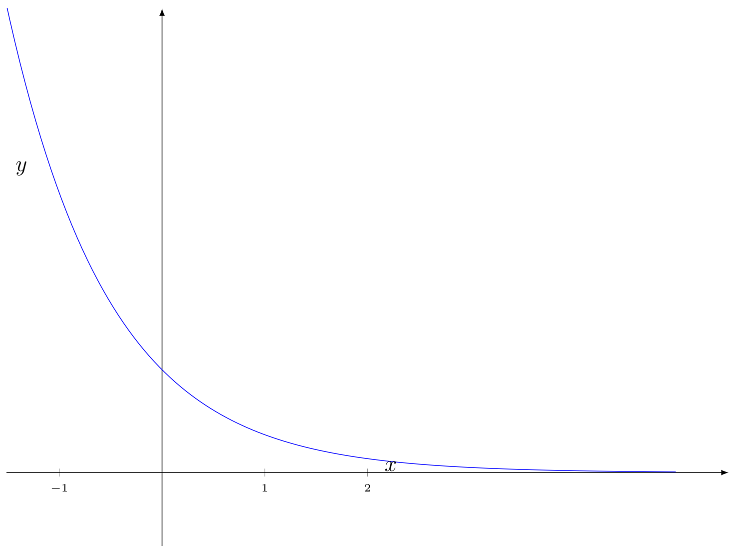
Problem 1. Consider the following claim:

$$\lim_{x\to+\infty} e^{-x} = 0.$$

(1). Make a sketch of the graph of the function $f(x) = e^{-x}$ and determine, based on that graph, if the above claim makes sense. (An intuitive conclusion without rigorous proof is sufficient for full credit.)

Space for your solution:

As x approaches $+\infty$, the value of e^{-x} does indeed approach 0, thus making it intuitively clear that the statement about limit is correct:



(2). Using the definition of limit, express the above claim as a precise statement of predicate logic.

Space for your solution:

Specializing the general definition of limit at $+\infty$:

$$\left(\lim_{x\to+\infty}\Big(f(x)\Big) = L\right) \quad\Leftrightarrow\quad \left(\forall\varepsilon > 0\left(\exists M > 0\left(\forall x\left(x > M \Rightarrow \Big|f(x) - L\Big| < \varepsilon\right)\right)\right)\right)$$

to our case, we get:

$$\left(\lim_{x\to+\infty}\left(e^{-x}\right) = 0\right) \quad\Leftrightarrow\quad \left(\forall\varepsilon > 0\left(\exists M > 0\left(\forall x\left(x > M \Rightarrow \left|e^{-x}\right| < \varepsilon\right)\right)\right)\right).$$

(3). Using the rigorous and precise definition of the limit from the previous subproblem, prove or disprove the statement

$$\lim_{x \to +\infty} e^{-x} = 0.$$

Space for your solution:

$$\left(\lim_{x \to -\infty} \left(e^{-x} \right) = 0 \right) \Leftarrow \boxed{\text{definition of limit from the previous subproblem}} \Rightarrow$$

$$\left(\forall \varepsilon > 0 \left(\exists M > 0 \left(\forall x \left(x > M \Rightarrow \left| e^{-x} \right| < \varepsilon \right) \right) \right) \right) \Leftarrow \boxed{e^{-x} > 0} \Rightarrow$$

$$\left(\forall \varepsilon > 0 \left(\exists M > 0 \left(\forall x \left(x > M \Rightarrow e^{-x} < \varepsilon \right) \right) \right) \right) \Leftarrow \boxed{\text{apply ln to both sides}} \Rightarrow$$

$$\left(\forall \varepsilon > 0 \left(\exists M > 0 \left(\forall x \left(x > M \Rightarrow -x < \ln(\varepsilon) \right) \right) \right) \right) \Leftarrow \boxed{\text{opposite of both sides}} \Rightarrow$$

$$\left(\forall \varepsilon > 0 \left(\exists M > 0 \left(\forall x \left(x > M \Rightarrow x > -\ln(\varepsilon) \right) \right) \right) \right) \Leftarrow \boxed{\text{exclude } x} \Rightarrow$$

$$\left(\forall \varepsilon > 0 \left(\exists M > 0 \left(M \geq -\ln(\varepsilon) \right) \right) \right) \Leftarrow \boxed{\text{explicit witness for existential quantifier}} \Rightarrow$$

$$\left(\forall \varepsilon > 0 \left(\exists M > 0 \left(M = \max(-\ln(\varepsilon), 1) \right) \right) \right) \Leftrightarrow \text{TRUE.}$$

(In the above definition of the witness, we use the maximum of the two values to address for the possibility of $-\ln(\varepsilon)$ being negative, which happens for $\varepsilon \geq 1$.)

Problem 2. Relying on the limit of composition theorem and the knowledge of asymptotic behavior of e^x, find

$$\lim_{x\to+\infty} \frac{e^{x^2+5}}{e^{2x^2+3} - e^{2x^2-1}}.$$

(You must show how the result follows from the things you may rely on; the answer alone is not sufficient.)

Space for your solution:

The denominator of the fraction is the difference of two exponents, each approaching $+\infty$ as x goes to $+\infty$. In other words, the denominator is the case of $(+\infty) - (+\infty)$ indeterminancy, which requires factoring out the main term of each $+\infty$.

$$\lim_{x\to+\infty} \frac{e^{x^2+5}}{e^{2x^2+3} - e^{2x^2-1}} = \lim_{x\to+\infty} \frac{e^{x^2} e^5}{e^{2x^2}(e^3 - e^{-1})} = \lim_{x\to+\infty} \frac{e^{x^2}}{e^{2x^2}} \cdot \frac{e^5}{e^3 - e^{-1}} =$$

$$\lim_{x\to+\infty} \frac{1}{e^{x^2}} \cdot \frac{e^5}{e^3 - e^{-1}} \boxed{=\text{take multiplicative constant out of the limit}=} \frac{e^5}{e^3-e^{-1}} \cdot \lim_{x\to+\infty} \frac{1}{e^{x^2}}$$

$$= \frac{e^5}{e^3 - e^{-1}} \cdot \lim_{x\to+\infty} e^{-x^2} \boxed{=\text{asymptotic behavior of } e^x=} \frac{e^5}{e^3-e^{-1}} \cdot 0 = 0.$$

Problem 3. Consider the function with range $\mathbb{R}$ defined as $f(x) = \frac{1}{x}$ on the maximum set of real numbers x for which this formula makes sense.

(1). What is the domain of the function f?

Space for your solution:

The formula that defines f makes sense for all non-zero values of the denominator, therefore

$$\texttt{Dom } f = \{x \in \mathbb{R} : x \neq 0\} = (-\infty, 0) \cap (0, +\infty).$$

(2). Is the function $f(x)$ continuous at $x = 5$? Why? (Making intuitively obvious statements about certain limits without proof is acceptable.)

Space for your solution:

Yes, because $\lim_{x\to 5} f(x) = \dfrac{1}{5} = f(5)$.

(3). Is the function $f(x)$ continuous at $x = 5$? Prove your answer starting from basic principles. (Making intuitively obvious statements without poof is not acceptable.)

Space for your solution:

$\lim_{x\to 5} \frac{1}{x} = \frac{1}{5} \Leftarrow$ definition of limit $\Rightarrow$

$\forall \varepsilon > 0 \left(\exists \delta > 0 \left(\forall x \in \mathbb{R} \; (0 < |x-5| < \delta \;\Rightarrow\; |\frac{1}{x} - \frac{1}{5}| < \varepsilon)\right)\right) \Leftarrow$ common denominator $\Rightarrow$

$\forall \varepsilon > 0 \left(\exists \delta > 0 \left(\forall x \in \mathbb{R} \; (0 < |x-5| < \delta \;\Rightarrow\; |\frac{5-x}{5x}| < \varepsilon)\right)\right) \Leftarrow$ distribute abs. value $\Rightarrow$

$\forall \varepsilon > 0 \left(\exists \delta > 0 \left(\forall x \in \mathbb{R} \; (0 < |x-5| < \delta \;\Rightarrow\; \frac{|5-x|}{|5x|} < \varepsilon)\right)\right) \Leftarrow$ multiply by $|5x|$ $\Rightarrow$

$\forall \varepsilon > 0 \left(\exists \delta > 0 \left(\forall x \in \mathbb{R} \; (0 < |x-5| < \delta \;\Rightarrow\; |5-x| < \varepsilon \cdot |5x|)\right)\right)$

Now take any positive $\delta < \min(1, 20\varepsilon)$. On one hand, since $\delta < 20\varepsilon$, we have that $|x-5| < 20\varepsilon$. On the other hand, since $\delta < 1$, we can be sure that $x > 4$ and therefore $\varepsilon \cdot |5x| > 20\varepsilon$. Combining these two facts together, we get

$$|x-5| < 20\varepsilon < \varepsilon \cdot |5x|.$$

(4). Is the function $f(x)$ continuous at $x = 0$? Why?

Space for your solution:

No, because $f(x)$ is not defined at $x = 0$.

(5). Is the function $f(x)$ continuous on the set of all real numbers?

Space for your solution:

No, because the function $f(x)$ is not defined on the whole set of real numbers.

(6). Is the function f continuous?

Space for your solution:

Yes.

Similar to subproblem (2), we have that $\lim_{x\to x_0} f(x) = f(x_0)$ for every non-zero value of x_0, making the function f continuous at every point x_0 in its domain. Thus, by definition of continutity, the function f itself is continuous.

(7). Classify all real numbers as points continuity of f, or points of discontinuity of f (i.e. the points where the function is defined, but is not continuous), or removable singularities of f, or significant singularities of f, or none of the above.

Space for your solution:

All points of $\mathtt{Dom}\ f = \{x \in \mathbb{R} : x \neq 0\}$ are points of continuity of f, and $x = 0$ is a significant singularity, since f does not have any limit in $\mathbb{R}$ at that point.

(8). Suppose that $\mathbb{R}$, the range of the function f, is compactified with infinity: $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Classify all real numbers as in the previous problem. (Strictly speaking, the function we are talking about in this sub-problem is not the f, but a new function $\overline{f}$, defined the same way as f, except as one having the range $\overline{\mathbb{R}}$ instead of $\mathbb{R}$.)

Space for your solution:

All points of $\mathtt{Dom}\ f = \{x \in \mathbb{R} : x \neq 0\}$ are regular points of f, and $x = 0$ is a removable singularity. Indeed, $\lim_{x \to 0} f(x) = \dfrac{1}{x} = \infty$. Therefore we can remove the singularity of f at $x = 0$ by defining

$$\hat{f}(x) = \begin{cases} f(x), & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{cases}$$

Then $\hat{f}$ is a continuous function with domain $\mathbb{R}$ and range $\mathbb{R} \cup \{\infty\}$.

Problem 4. Find $\frac{\mathtt{d}}{\mathtt{d}x} \frac{4x-5}{x^3-1}$.

Space for your solution:

$$\frac{\mathtt{d}}{\mathtt{d}x} \frac{4x-5}{x^3-1} =$$

$$\frac{\left(\frac{\mathtt{d}}{\mathtt{d}x}(4x-5)\right) \cdot (x^3-1) - (4x-5) \cdot \left(\frac{\mathtt{d}}{\mathtt{d}x}(x^3-1)\right)}{(x^3-1)^2} =$$

$$\frac{4 \cdot (x^3-1) - (4x-5) \cdot (3x^2)}{(x^3-1)^2} = \frac{4x^3 - 4 - (12x^3 - 15x^2)}{x^6 - 2x^3 + 1} = \frac{-8x^3 + 15x^2 - 4}{x^6 - 2x^3 + 1}$$

Problem 5. Find

$$\frac{\mathtt{d}}{\mathtt{d}x}\arcsin\left(\frac{x}{\sqrt{x}+\ln x}\right).$$

Space for your solution:

$$\frac{\mathtt{d}}{\mathtt{d}x}\arcsin\left(\frac{x}{\sqrt{x}+\ln x}\right) = \boxed{\text{chain rule}} = \frac{\mathtt{d}\arcsin\left(\frac{x}{\sqrt{x}+\ln x}\right)}{\mathtt{d}\left(\frac{x}{\sqrt{x}+\ln x}\right)} \cdot \frac{\mathtt{d}\left(\frac{x}{\sqrt{x}+\ln x}\right)}{\mathtt{d}x}$$

$$= \boxed{\text{derivative of arcsin and of a quotient}} =$$

$$= \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{x}+\ln x}\right)^2}} \cdot \frac{\left(\frac{\mathtt{d}}{\mathtt{d}x}x\right)\cdot\left(\sqrt{x}+\ln x\right) - (x)\cdot\left(\frac{\mathtt{d}}{\mathtt{d}x}\left(\sqrt{x}+\ln x\right)\right)}{\left(\sqrt{x}+\ln x\right)^2} =$$

$$= \boxed{\text{derivative of } x \text{ and of a sum}} =$$

$$= \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{x}+\ln x}\right)^2}} \cdot \frac{\left(\sqrt{x}+\ln x\right) - x\cdot\left(\frac{\mathtt{d}}{\mathtt{d}x}\left(\sqrt{x}\right) + \frac{\mathtt{d}}{\mathtt{d}x}\left(\ln x\right)\right)}{\left(\sqrt{x}+\ln x\right)^2} =$$

$$= \boxed{\text{derivative of the square root and of the logarithm}} =$$

$$= \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{x}+\ln x}\right)^2}} \cdot \frac{\left(\sqrt{x}+\ln x\right) - x\cdot\left(\frac{1}{2\sqrt{x}} + \frac{1}{x}\right)}{\left(\sqrt{x}+\ln x\right)^2}.$$