Suffolk County Community College

Michael J. Grant Campus

Department of Mathematics

Wednesday, December 11, 2024 (returned Monday, December 16, 2024)

MAT 141
Calculus with Analytic Geometry I

Final Exam

Instructor:

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Please print the requested information in the spaces provided:

Student:

Name:

Student Id:

Email: *include to receive the final grade via email* ONLY *if you are not getting email updates*

- *Notes and books are permitted on this exam.*
- *Graphing calculators, smartwatches, computers, cell phones and any other communication-capable devices are prohibited. Their mere presence in the open (even without use) is a sufficient reason for an immediate dismissal from this exam with a failing grade.*
- *You will not receive full credit if there is no work shown, even if you have the right answer. Please don't attach additional pieces of paper: if you run out of space, please ask for another blank final.*

Problem 1. Consider the following claim:

$$\lim_{x \to -\infty} \frac{1}{x} = 0.$$

(1). Make a sketch of the graph of the function $f(x) = \frac{1}{x}$ and determine, based on that graph, if the above claim makes sense. (An intuitive conclusion without rigorous proof is sufficient for full credit.)

Space for your solution:

(2). Using the definition of limit, express the above claim as a precise statement of predicate logic.

Space for your solution:

(3). Using the rigorous and precise definition of the limit from the previous subproblem, prove or disprove the statement

$$\lim_{x \to -\infty} \frac{1}{x} = 0.$$

Space for your solution:

Problem 2. Relying on the limit of composition theorem and the knowledge of asymptotic behavior of $\frac{1}{x}$ at $-\infty$ find

$$\lim_{x\to-\infty} \frac{5x^7 - 3x^5 + 2x^3}{6x^7 + 2x + 1}.$$

(You must show how the result follows from the things you may rely on; the answer alone is not sufficient.)

Space for your solution:

Problem 3. Find $\frac{\mathtt{d}\,\mathrm{e}^{\sin(x)+\cos\left(x^2\right)}}{\mathtt{d}x}$.

Space for your solution:

Problem 4. Find $\frac{\mathtt{d}}{\mathtt{d}x} \operatorname{arccot}\left(\frac{e^x}{x+\sqrt{x}}\right)$.

Space for your solution:

Problem 5. Find the maximum value and the point(s) of maximum of the function $f(x) = 3x^3 - x$ on the interval $[-1, 0]$, if they exist.

Space for your solution:

Problem 6. Suppose B is a celestial body of mass M, and x is a position in space, somewhere away from the B. A vector v starting at x will be called *an escape velocity* of B at x if any projectile starting at x with the initial velocity v will leave the gravitational pull of B. (We assume that once the projectile is given its initial velocity, it moves inertially without receiving any external impulse or force, other than the force of gravity from B.) It turns out that for any such B and x, escape velocities always exist.

The shortest escape velocity v pointing radially away from B is called *the escape velocity* of B at x. The length of the escape velocity at x depends only on the mass M of B and the distance r_0 of x from the B. (It does not depend on the mass m of the projectile, or the specific position x.) Since we know the direction of the escape velocity, we will only concern ourselves with its length (but will still call it *the escape velocity* rather than *escape speed*).

In this problem we will find the escape velocity of the Solar System for a projectile launched from a low Earth orbit.

(1). The Newton's law of gravity $F = G \cdot \frac{M \cdot m}{r^2}$ describes the force of gravity F (in Newtons) in terms of the gravitational constant $G = 6.67 \cdot 10^{-11} \cdot \frac{\mathrm{m}^3}{\mathrm{kg} \cdot \mathrm{s}^2}$, the masses M and m of the two attracting bodies (in kilograms), and the distance r between them (in meters). Compute the work $W = \int_{r_0}^{+\infty} F(r)\, dr$ done against the force of gravity F of the celestial object B when moving a projectile of mass m radially away from B starting a distance r_0 from the B to infinity.

Space for your solution:

(2). As the projectile escapes the gravitational pull of B, it uses up its kinetic energy $E = \frac{m \cdot v^2}{2}$ to do work against the force of gravity F. Thus in order for the projectile's speed to be sufficient for escaping the gravitational pull of B, the initial kinetic energy E of the projectile must equal to the work W computed above. Use this and the formula for W found in the previous sub-problem to derive the formula for the escape velocity v [1].

Space for your solution:

[1] Since we know the direction of the escape velocity — away from the celestial object — what we are really looking for is the speed, i.e. the length of the velocity.

(3). Assume that the mass of the Solar System equals the mass of the Sun and is positioned at the center of the Sun[2]. The distance from Earth to the Sun is $1.49597870700 \cdot 10^{11} \cdot \mathrm{m}$, and the mass of the Sun is $1.989 \cdot 10^{30} \cdot \mathrm{kg}$. Compute the minimal speed of a projectile launched from a near Earth orbit, sufficient for escaping the gravitational pull of the Solar System. Assume that the distance of the projectile from the Solar System is the distance from Earth to the Sun.

Space for your solution:

[2] It is not much of an error, since the Sun accounts for about 99.8% of the total mass of the Solar System.