Suffolk County Community College

Michael J. Grant Campus

Department of Mathematics

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MAT 141
Calculus with Analytic Geometry I

Final Exam: Solutions and Answers

Instructor:

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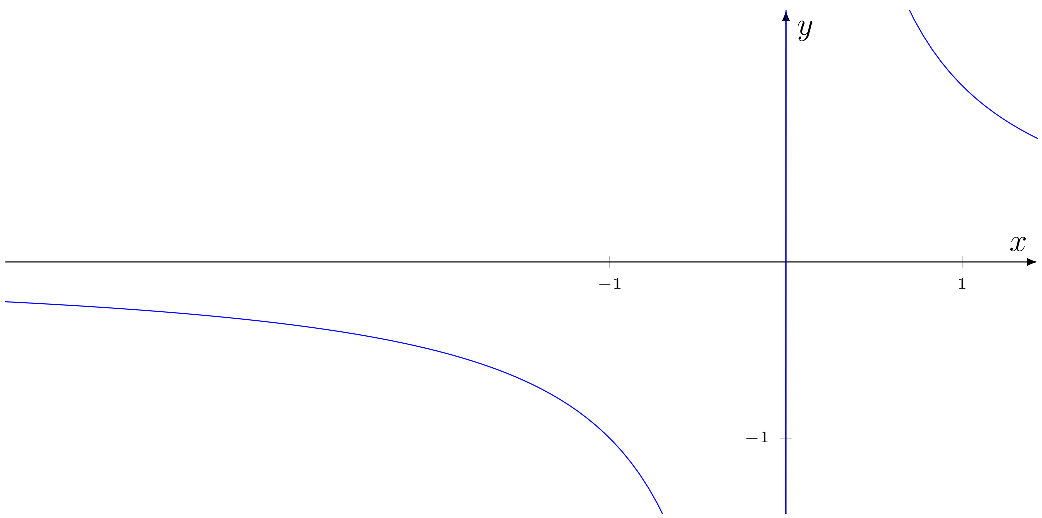
Problem 1. Consider the following claim:

$$\lim_{x \to -\infty} \frac{1}{x} = 0.$$

(1). Make a sketch of the graph of the function $f(x) = \frac{1}{x}$ and determine, based on that graph, if the above claim makes sense. (An intuitive conclusion without rigorous proof is sufficient for full credit.)

Space for your solution:

As x approaches $-\infty$, the value of $\frac{1}{x}$ does indeed approach 0, thus making it intuitively clear that the statement about limit is correct:



(2). Using the definition of limit, express the above claim as a precise statement of predicate logic.

Space for your solution:

Specializing the general definition of limit at $-\infty$:

$$\left(\lim_{x\to-\infty}\Big(f(x)\Big) = L\right) \quad \Leftrightarrow \quad \left(\forall\varepsilon > 0 \left(\exists M > 0 \left(\forall x \left(x < -M \Rightarrow \Big|f(x) - L\Big| < \varepsilon\right)\right)\right)\right)$$

to our case, we get:

$$\left(\lim_{x\to-\infty}\left(\frac{1}{x}\right) = 0\right) \quad \Leftrightarrow \quad \left(\forall\varepsilon > 0 \left(\exists M > 0 \left(\forall x \left(x < -M \Rightarrow \left|\tfrac{1}{x}\right| < \varepsilon\right)\right)\right)\right).$$

(3). Using the rigorous and precise definition of the limit from the previous subproblem, prove or disprove the statement

$$\lim_{x \to -\infty} \frac{1}{x} = 0.$$

Space for your solution:

$$\left(\lim_{x \to -\infty} \left(\frac{1}{x} \right) = 0 \right) \Leftarrow\!\boxed{\text{definition of limit from the previous subproblem}}\!\Rightarrow$$

$$\left(\forall \varepsilon > 0 \left(\exists M > 0 \left(\forall x \left(x < -M \Rightarrow \left| \tfrac{1}{x} \right| < \varepsilon \right) \right) \right) \right) \Leftarrow\!\boxed{\tfrac{1}{x} < 0 \text{ since } x < -M < 0}\!\Rightarrow$$

$$\left(\forall \varepsilon > 0 \left(\exists M > 0 \left(\forall x \left(x < -M \Rightarrow -\tfrac{1}{x} < \varepsilon \right) \right) \right) \right) \Leftarrow\!\boxed{\text{multiply both sides by } x}\!\Rightarrow$$

$$\left(\forall \varepsilon > 0 \left(\exists M > 0 \left(\forall x \left(x < -M \Rightarrow -1 > x \cdot \varepsilon \right) \right) \right) \right) \Leftarrow\!\boxed{\text{divide both sides by } \varepsilon}\!\Rightarrow$$

$$\left(\forall \varepsilon > 0 \left(\exists M > 0 \left(\forall x \left(x < -M \Rightarrow -\tfrac{1}{\varepsilon} > x \right) \right) \right) \right) \Leftarrow\!\boxed{\text{reverse the inequality}}\!\Rightarrow$$

$$\left(\forall \varepsilon > 0 \left(\exists M > 0 \left(\forall x \left(x < -M \Rightarrow x < -\tfrac{1}{\varepsilon} \right) \right) \right) \right) \Leftarrow\!\boxed{\text{exclude } x}\!\Rightarrow$$

$$\left(\forall \varepsilon > 0 \left(\exists M > 0 \left(-M \leq -\tfrac{1}{\varepsilon} \right) \right) \right) \Leftarrow\!\boxed{\text{explicit witness for existential quantifier}}\!\Rightarrow$$

$$\left(\forall \varepsilon > 0 \left(\exists M > 0 \left(M = \tfrac{1}{\varepsilon} \right) \right) \right) \quad \Leftrightarrow \text{ TRUE.}$$

Problem 2. Relying on the limit of composition theorem and the knowledge of asymptotic behavior of $\frac{1}{x}$ at $-\infty$ find

$$\lim_{x\to-\infty} \frac{5x^7 - 3x^5 + 2x^3}{6x^7 + 2x + 1}.$$

(You must show how the result follows from the things you may rely on; the answer alone is not sufficient.)

Space for your solution:

This limit is the case of $\frac{\infty}{\infty}$ indeterminacy. Therefore, finding the limit requires cancellation of the main terms of each ∞:

$$\lim_{x\to-\infty} \frac{5x^7 - 3x^5 + 2x^3}{6x^7 + 2x + 1} =$$

$$= \boxed{\text{factor out the main terms of the numerator and denominator}} =$$

$$\lim_{x\to-\infty} \frac{x^7(5 - \frac{3}{x^2} + \frac{2}{x^4})}{x^7(6 + \frac{2}{x^6} + \frac{1}{x^7})} =$$

$$= \boxed{\text{cancel the main terms}} =$$

$$\lim_{x\to-\infty} \frac{5 - \frac{3}{x^2} + \frac{2}{x^4}}{6 + \frac{2}{x^6} + \frac{1}{x^7}} = \lim_{x\to-\infty} \frac{5 - 3\cdot(\frac{1}{x})^2 + 2\cdot(\frac{1}{x})^4}{6 + 2\cdot(\frac{1}{x})^6 + (\frac{1}{x})^7} =$$

$$= \boxed{\text{limit of composition theorem}} =$$

$$\frac{5 - 3\cdot\left(\lim\limits_{x\to-\infty}\frac{1}{x}\right)^2 + 2\cdot\left(\lim\limits_{x\to-\infty}\frac{1}{x}\right)^4}{6 + 2\cdot\left(\lim\limits_{x\to-\infty}\frac{1}{x}\right)^6 + \left(\lim\limits_{x\to-\infty}\frac{1}{x}\right)^7}$$

$$= \boxed{\text{asymptotic behavior of } \frac{1}{x} \text{ as } x\to-\infty} =$$

$$\frac{5 - 3\cdot 0 + 2\cdot 0}{6 + 2\cdot 0 + 0} = \frac{5}{6}.$$

Problem 3. Find $\frac{\mathtt{d}\, \mathrm{e}^{\sin(x)+\cos\left(x^2\right)}}{\mathtt{d}x}$.

Space for your solution:

$$\frac{\mathtt{d}\, \mathrm{e}^{\sin(x)+\cos\left(x^2\right)}}{\mathtt{d}x} = \boxed{\text{chain rule}} =$$

$$\frac{\mathtt{d}\, \mathrm{e}^{\sin(x)+\cos\left(x^2\right)}}{\mathtt{d}\Big(\sin(x)+\cos\left(x^2\right)\Big)} \cdot \frac{\mathtt{d}\Big(\sin(x)+\cos\left(x^2\right)\Big)}{\mathtt{d}x}$$

$$= \boxed{\text{derivative of exponent, derivative of sum}} =$$

$$\mathrm{e}^{\sin(x)+\cos\left(x^2\right)} \cdot \left(\frac{\mathtt{d}\sin(x)}{\mathtt{d}x} + \frac{\mathtt{d}\cos\left(x^2\right)}{\mathtt{d}x}\right)$$

$$= \boxed{\text{derivative of sin, chain rule}} =$$

$$\mathrm{e}^{\sin(x)+\cos\left(x^2\right)} \cdot \left(\cos(x) + \frac{\mathtt{d}\cos\left(x^2\right)}{\mathtt{d}\left(x^2\right)} \cdot \frac{\mathtt{d}\left(x^2\right)}{\mathtt{d}x}\right)$$

$$= \boxed{\text{derivative of cosine, derivative of power}} =$$

$$\mathrm{e}^{\sin(x)+\cos\left(x^2\right)} \cdot \Big(\cos(x) - 2x \cdot \sin\left(x^2\right)\Big).$$

Problem 4. Find $\frac{\mathtt{d}}{\mathtt{d}x}\operatorname{arccot}\left(\frac{e^x}{x+\sqrt{x}}\right)$.

Space for your solution:

$$\frac{\mathtt{d}}{\mathtt{d}x}\operatorname{arccot}\left(\frac{e^x}{x+\sqrt{x}}\right) = \boxed{\text{chain rule}} = \frac{\mathtt{d}\operatorname{arccot}\left(\frac{e^x}{x+\sqrt{x}}\right)}{\mathtt{d}\left(\frac{e^x}{x+\sqrt{x}}\right)} \cdot \frac{\mathtt{d}\left(\frac{e^x}{x+\sqrt{x}}\right)}{\mathtt{d}x}$$

$$= \boxed{\text{derivative of arccot and of quotient}} =$$

$$= -\frac{1}{1+\left(\frac{e^x}{x+\sqrt{x}}\right)^2} \cdot \frac{\left(\frac{\mathtt{d}}{\mathtt{d}x}e^x\right)\cdot\left(x+\sqrt{x}\right) - (e^x)\cdot\left(\frac{\mathtt{d}}{\mathtt{d}x}\left(x+\sqrt{x}\right)\right)}{\left(x+\sqrt{x}\right)^2} =$$

$$= \boxed{\text{derivative of exponent and of sum}} =$$

$$= -\frac{1}{1+\left(\frac{e^x}{x+\sqrt{x}}\right)^2} \cdot \frac{e^x\cdot\left(x+\sqrt{x}\right) - e^x\cdot\left(\frac{\mathtt{d}}{\mathtt{d}x}x + \frac{\mathtt{d}}{\mathtt{d}x}\sqrt{x}\right)}{\left(x+\sqrt{x}\right)^2} =$$

$$= \boxed{\text{derivative of } x \text{ and of } \sqrt{x}} =$$

$$= -\frac{1}{1+\left(\frac{e^x}{x+\sqrt{x}}\right)^2} \cdot \frac{e^x\cdot\left(x+\sqrt{x}\right) - e^x\cdot\left(1+\frac{1}{2\sqrt{x}}\right)}{\left(x+\sqrt{x}\right)^2}.$$

Problem 5. Find the maximum value and the point(s) of maximum of the function $f(x) = 3x^3 - x$ on the interval $[-1, 0]$, if they exist.

Space for your solution:

The function $f(x)$ is a polynomial defined on a closed interval, Thus, being a continuous function on a compact set, it has a maximum and a minimum, as well as point(s) of maximum and minimum, on that set.

All internal points with non-zero derivative can be ruled out: they are neither the points of maximum, nor the points of minimum, of the function in question. All other points, namely:

- the end points of the interval,
- the points where the derivative $\frac{\mathtt{d}}{\mathtt{d}x} f(x)$ is undefined, and finally
- the points where the derivative $\frac{\mathtt{d}}{\mathtt{d}x} f(x)$ is defined and equals to zero,

will be examined one by one to identify the point(s) of maximum among them.

In our case,

- the end points of the interval are $x_1 = -1$ and $x_2 = 0$,
- the derivative $\frac{\mathtt{d}\left(3x^3 - x\right)}{\mathtt{d}x} = 9x^2 - 1$ is defined everywhere on the interval $[-1, 0]$, and
- solving for the points of the interval $[-1, 3]$ where the derivative is zero: $\begin{cases} 9x^2 - 1 = 0 \\ -1 \le x \le 0 \end{cases} \Leftrightarrow \begin{cases} x^2 = \frac{1}{9} \\ -1 \le x \le 0 \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} x = \frac{1}{3} \\ x = -\frac{1}{3} \end{array} \right. \\ -1 \le x \le 0 \end{cases} \Leftrightarrow x = -\frac{1}{3}$, we get the last suspect $x_3 = -\frac{1}{3}$.

Thus the complete list of suspects is: $x_1 = -1$, $x_2 = 0$, $x_3 = -\frac{1}{3}$. The values of the function $f(x)$ at these points are:

$$f(-1) = -2,\ f(0) = 0,\ f\left(-\frac{1}{3}\right) = \frac{2}{9}.$$

The maximum of those numbers is $f\left(-\frac{1}{3}\right) = \frac{2}{9}$. Therefore the *point of maximum* is $x = -\frac{1}{3}$ and the *maximum value* is $\frac{2}{9}$.

Problem 6. Suppose B is a celestial body of mass M, and x is a position in space, somewhere away from the B. A vector v starting at x will be called *an escape velocity* of B at x if any projectile starting at x with the initial velocity v will leave the gravitational pull of B. (We assume that once the projectile is given its initial velocity, it moves inertially without receiving any external impulse or force, other than the force of gravity from B.) It turns out that for any such B and x, escape velocities always exist.

The shortest escape velocity v pointing radially away from B is called *the escape velocity* of B at x. The length of the escape velocity at x depends only on the mass M of B and the distance r_0 of x from the B. (It does not depend on the mass m of the projectile, or the specific position x.) Since we know the direction of the escape velocity, we will only concern ourselves with its length (but will still call it *the escape velocity* rather than *escape speed*).

In this problem we will find the escape velocity of the Solar System for a projectile launched from a low Earth orbit.

(1). The Newton's law of gravity $F = G \cdot \frac{M \cdot m}{r^2}$ describes the force of gravity F (in Newtons) in terms of the gravitational constant $G = 6.67 \cdot 10^{-11} \cdot \frac{\mathrm{m}^3}{\mathrm{kg} \cdot \mathrm{s}^2}$, the masses M and m of the two attracting bodies (in kilograms), and the distance r between them (in meters). Compute the work $W = \int_{r_0}^{+\infty} F(r)\, dr$ done against the force of gravity F of the celestial object B when moving a projectile of mass m radially away from B starting a distance r_0 from the B to infinity.

Space for your solution:

$$W = \int_{r_0}^{+\infty} F(r)\, dr = \int_{r_0}^{+\infty} G \cdot \frac{M \cdot m}{r^2}\, dr = G \cdot M \cdot m \int_{r_0}^{+\infty} \frac{dr}{r^2} = G \cdot M \cdot m \int_{r_0}^{+\infty} d\left(-\frac{1}{r}\right) =$$

$$G \cdot M \cdot m \cdot \left(-\frac{1}{r}\right)\Bigg|_{r_0}^{+\infty} = G \cdot M \cdot m \cdot \left(-\frac{1}{+\infty} + \frac{1}{r_0}\right) = \frac{G \cdot M \cdot m}{r_0}.$$

(2). As the projectile escapes the gravitational pull of B, it uses up its kinetic energy $E = \frac{m \cdot v^2}{2}$ to do work against the force of gravity F. Thus in order for the projectile's speed to be sufficient for escaping the gravitational pull of B, the initial kinetic energy E of the projectile must equal to the work W computed above. Use this and the formula for W found in the previous sub-problem to derive the formula for the escape velocity v [1].

Space for your solution:

Solving the equation $W = E$ for the speed v, we get:

$W = E \Leftarrow$ $\boxed{\text{use the expression for } W \text{ from the previous problem}} \Rightarrow$

$\frac{G \cdot M \cdot m}{r_0} = \frac{m \cdot v^2}{2} \quad \Leftrightarrow \quad v = \sqrt{\frac{2GM}{r_0}}.$

[1] Since we know the direction of the escape velocity — away from the celestial object — what we are really looking for is the speed, i.e. the length of the velocity.

(3). Assume that the mass of the Solar System equals the mass of the Sun and is positioned at the center of the Sun[2]. The distance from Earth to the Sun is $1.49597870700 \cdot 10^{11} \cdot \mathrm{m}$, and the mass of the Sun is $1.989 \cdot 10^{30} \cdot \mathrm{kg}$. Compute the minimal speed of a projectile launched from a near Earth orbit, sufficient for escaping the gravitational pull of the Solar System. Assume that the distance of the projectile from the Solar System is the distance from Earth to the Sun.

Space for your solution:

$$v = \sqrt{\frac{2GM}{r_0}} = \sqrt{\frac{2 \cdot 6.67 \cdot 10^{-11} \cdot \frac{\mathrm{m}^3}{\mathrm{kg} \cdot \mathrm{s}^2} \cdot 1.989 \cdot 10^{30} \cdot \mathrm{kg}}{149,597,870,700 \cdot \mathrm{m}}} =$$

$$\sqrt{\frac{2 \cdot 6.67 \cdot 10^{-11} \cdot \frac{\mathrm{m}^3}{\mathrm{kg} \cdot \mathrm{s}^2} \cdot 1.989 \cdot 10^{30} \cdot \mathrm{kg}}{1.49597870700 \cdot 10^{11} \cdot \mathrm{m}}} = \sqrt{\frac{2 \cdot 6.67 \cdot 1.989}{1.49597870700}} \cdot 10^4 \cdot \frac{\mathrm{m}}{\mathrm{s}} \approx$$

$$42,115 \cdot \frac{\mathrm{m}}{\mathrm{s}} \approx 42 \cdot \frac{\mathrm{km}}{\mathrm{s}}.$$

[2] It is not much of an error, since the Sun accounts for about 99.8% of the total mass of the Solar System.