Suffolk County Community College

Michael J. Grant Campus

Department of Mathematics

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MAT 142
Calculus with Analytic Geometry II

Final Exam: Solutions and Answers

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Problem 1. Compute the integral

$$\int e^{\sqrt{x}} \ \mathtt{d}x.$$

Space for your solution:

First step is to rationalize the radical (page 25 of the notes). $\int e^{\sqrt{x}} \ \mathtt{d}x = \boxed{u = \sqrt{x}} = \int e^u \ \mathtt{d}u^2 = \int 2u \ e^u \ \mathtt{d}u$. Now we can use integration by parts (page 15 of the notes). The function $2u \ e^u$ that we have to integrate is a product of $2u$ and e^u. Differentiation will turn $2u$ into 2 which is definitely a simpler function than $2u$. Integration will turn e^u to e^u again — no more complicated than it was. In other words,

$$\int 2u \ e^u \ \mathtt{d}u = \int 2u \ \mathtt{d}e^u = 2u \ e^u - \int e^u \ \mathtt{d}(2u) =$$
$$2u \ e^u - 2\int e^u \ \mathtt{d}u = 2u \ e^u - 2e^u + C = 2\sqrt{x} \ e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.$$

Problem 2. Compute the integral

$$\int x \ \cos(6x^2 - 1) \ \mathtt{d}x.$$

Space for your solution:

$$\int x \ \cos(6x^2 - 1) \ \mathtt{d}x = \boxed{\mathtt{d}(6x^2 - 1) = 12x \ \mathtt{d}x} =$$
$$\frac{1}{12}\int \cos(6x^2 - 1) \ \mathtt{d}(6x^2 - 1) = \frac{1}{12}\sin(6x^2 - 1) + C$$

Problem 3. Compute the integral

$$\int \frac{x^3 - 8x^2 + 21x - 19}{x^2 - 6x + 9} \; \mathtt{d}x.$$

Space for your solution:

1. Divide the polynomial $x^3 - 8x^2 + 21x - 19$ by the polynomial $x^2 - 6x + 9$.

$$\begin{array}{rl} & x - 2 \\ x^2 - 6x + 9 \;\big) & \overline{x^3 - 8x^2 + 21x - 19} \\ & \underline{x^3 - 6x^2 + 9x} \\ & -2x^2 + 12x - 19 \\ & \underline{-2x^2 + 12x - 18} \\ & -1 \end{array}$$

   The above long division means that

$$\int \frac{x^3 - 8x^2 + 21x - 19}{x^2 - 6x + 9} \; \mathtt{d}x = \int \left(x - 2 + \frac{-1}{x^2 - 6x + 9} \right) \; \mathtt{d}x.$$

2. Integrate $x - 2$:

$$\int (x - 2) \; \mathtt{d}x = \frac{(x-2)^2}{2} + C.$$

3. Using the method of partial fractions (page 21 of the notes), integrate $\frac{-1}{x^2 - 6x + 9}$:

$$\frac{-1}{x^2 - 6x + 9} = \frac{-1}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}.$$

   It is clear that our fraction itself happens to be a partial fraction (in other words, we can take $A = 0$, $B = -1$, and

$$\int \frac{-1}{(x-3)^2} \; \mathtt{d}x = -\int (x-3)^{-2} \; \mathtt{d}(x-3) = \frac{1}{x-3} + C.$$

4. Add the two integrals from above:

$$\int \frac{x^3 - 8x^2 + 21x - 19}{x^2 - 6x + 9} \; \mathtt{d}x = \frac{(x-2)^2}{2} + \frac{1}{x-3} + C.$$

Problem 4. Consider the function $f(t) = \ln(t)$.

(1). Which point (or points) x from the domain of the function f would be a good choice for the center of the Taylor polynomial for f and why?

Space for your solution:

To construct the Taylor polynomial of a function f, we need to know all $\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^i\Big|_{t=x} f(t)$. Compute the first column of the table from part (2), in other words, find all $\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^i f(t)$. After that job is done, it becomes clear that $x = 1$ is a good choice.

(2). Taking as $f(t) = \ln(t)$ and x being your choice in part (1), fill the table

i	$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^i f(t)$	$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^i\Big\|_{t=x} f(t)$
0		
1		
2		
3		
4		
5		
6		

Try to write your result in the form that would make it easy to see the general pattern for the i-th term of the Taylor polynomial.

Space for your solution:

i	$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^i f(t)$	$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^i\Big\|_{t=x} f(t)$
0	$\ln(t)$	0
1	$0! \cdot t^{-1}$	$0! = 1$
2	$-1! \cdot t^{-2}$	$-1! = -1$
3	$2! \cdot t^{-3}$	$2! = 2$
4	$-3! \cdot t^{-4}$	$-3! = -6$
5	$4! \cdot t^{-5}$	$4! = 24$
6	$-5! \cdot t^{-6}$	$-5! = -120$
...	...	...
i	$(-1)^{i-1} \cdot (i-1)! \cdot t^{-i}$	$(-1)^{i-1} \cdot (i-1)!$

(3). Compute the Taylor polynomial of degree 6 for the function f at one of the points you found in part (1). Be as explicit in your answer as you can. The explicit result is more important than a general pattern. In particular, don't use the $\sum$ notation for the sum.

Space for your solution:

$$\sum_{i=0}^{n} \frac{(\Delta x)^i}{i!} \left(\frac{\mathtt{d}}{\mathtt{d}t}\right)^i \Bigg|_{t=x} f(t) \boxed{f(t) = \ln(t),\ x = 1, \text{ and } n = 6} =$$

$$(\Delta x) - \frac{(\Delta x)^2}{2} + \frac{(\Delta x)^3}{3} - \frac{(\Delta x)^4}{4} + \frac{(\Delta x)^5}{5} - \frac{(\Delta x)^6}{6}.$$

(4). Compute the Taylor polynomial of degree n for the function f at one of the points you found in part (1). General pattern of the answer is more important here than being explicit. Use the $\sum$ notation for the sum.

Space for your solution:

$$\sum_{i=0}^{n} \frac{(\Delta x)^i}{i!} \left(\frac{\mathtt{d}}{\mathtt{d}t}\right)^i \Bigg|_{t=x} f(t) \boxed{f(t) = \ln(t),\ x = 1} = \sum_{i=1}^{n} \frac{(-1)^{i-1}(\Delta x)^i}{i}.$$

(5). Compute the error term for the Taylor polynomial of degree n for the function f at one of the points you found in part (1).

Space for your solution:

$$\frac{(\Delta x)^{n+1}}{(n+1)!}\int_{u=0}^{u=1}\left(\frac{\mathtt{d}}{\mathtt{d}t}\right)^{n+1} f\big(x+\Delta x\cdot(1-u)\big)\ \mathtt{d}u^{n+1} = \boxed{f(t)=\ln(t),\ x=1} =$$

$$\frac{(\Delta x)^{n+1}}{(n+1)!}\int_{u=0}^{u=1}\frac{(-1)^n n!}{\big(1+\Delta x\cdot(1-u)\big)^{n+1}}\cdot(n+1)\cdot u^n\cdot\ \mathtt{d}u$$

$$=(-1)^n\cdot(\Delta x)^{n+1}\int_{u=0}^{u=1}\frac{u^n\ \mathtt{d}u}{\big(1+\Delta x\cdot(1-u)\big)^{n+1}}.$$

(6). Find

- an estimate $E(n)$ of the error from part (5),
- conditions on the size of Δx that guarantee that

$$\lim_{n\to+\infty} E(n)=0$$

Make sure that your conditions cover the case needed for part (7).

Space for your solution:

$$\left|(-1)^n\cdot(\Delta x)^{n+1}\int_{u=0}^{u=1}\frac{u^n\ \mathtt{d}u}{\big(1+\Delta x\cdot(1-u)\big)^{n+1}}\right|$$

$$=|\Delta x|^{n+1}\cdot\left|\int_{u=0}^{u=1}\frac{u^n\ \mathtt{d}u}{\big(1+\Delta x\cdot(1-u)\big)^{n+1}}\right|\le|\Delta x|^{n+1}\cdot\max_{u\in[0,1]}\left|\frac{u^n}{\big(1+\Delta x\cdot(1-u)\big)^{n+1}}\right|$$

$$\le\frac{|\Delta x|^{n+1}}{\min\limits_{t\in[1,1+\Delta x]}|t|^{n+1}}=\frac{|\Delta x|^{n+1}}{\min\left\{|1|^{n+1},|1+\Delta x|^{n+1}]\right\}|}=\frac{|\Delta x|^{n+1}}{\min\left\{1,|1+\Delta x|^{n+1}]\right\}|}$$

$$=\boxed{\text{assume that (for example) } \Delta x\in[0,\tfrac{1}{2}]}=\tfrac{|\Delta x|^{n+1}}{1}=$$

$$=|\Delta x|^{n+1}\xrightarrow[n\to 0]{}0.$$

(Most certainly, this estimate of the error term is rather crude. Using a slightly more elaborate method, one can come up with an estimate that would work for any Δx smaller in size than 1.)

(7). Using your estimate of the error term from part (6), compute $\ln(1.2)$ with guaranteed accuracy of at least 0.01.

Space for your solution:

We need to find n, such that our error estimate $|\Delta x|^{n+1} < 0.01$. This particular $\Delta x = \frac{1}{5}$.
We have : $\left(\frac{1}{5}\right)^{n+1} < 0.01 \quad \Leftrightarrow \quad 5^{n+1} > 100 \quad \Leftrightarrow \quad n > 1$.
Therefore we can get the required accuracy by taking the Taylor polynomial of degree $n = 2$:

$$\ln(1.2) \approx (0.2) - \frac{(0.2)^2}{2} = 0.18$$

In other words, $\ln(1.2)$ is definitely between 0.17 and 0.19.