Suffolk County Community College

Michael J. Grant Campus

Department of Mathematics

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MAT 142
Calculus with Analytic Geometry II

Final Exam: Solutions and Answers

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Problem 1. Compute the integral

$$\int x \ \cos(6x^2 - 1) \ \mathtt{d}x.$$

Space for your solution:

We can use linear adjustment of substitution (page 19 of the notes):

$$\int x \ \cos(6x^2 - 1) \ \mathtt{d}x \ = \ \frac{1}{12}\int \cos(6x^2 - 1) \ \mathtt{d}(6x^2 - 1) \ = \ \frac{1}{12}\sin(6x^2 \ - \ 1) \ + \ C$$

Problem 2. Compute the integral

$$\int e^{\sqrt{x}} \ \mathtt{d}x.$$

Space for your solution:

Rationalize the radical (page 25 of the notes): $\int e^{\sqrt{x}} \ \mathtt{d}x = \boxed{u = \sqrt{x}} = \int e^u \ \mathtt{d}u^2 = \int 2u \ e^u \ \mathtt{d}u$. Then use integration by parts (page 15 of the notes): the function $2u \ e^u$ is a product of $2u$ and e^u. Differentiation will turn $2u$ into 2 which is definitely simpler than $2u$. Integration will turn e^u to e^u again — no more complicated than it was:

$$\int 2u \ e^u \ \mathtt{d}u = \int 2u \ \mathtt{d}e^u = 2u \ e^u - \int e^u \ \mathtt{d}(2u) =$$

$$2u \ e^u - 2\int e^u \ \mathtt{d}u = 2u \ e^u - 2e^u + C = 2\sqrt{x} \ e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.$$

Problem 3. Find the limit

$$\lim_{x\to 0} \frac{(\tan x) - x}{(\sin x) - x}.$$

Space for your solution:

$$\lim_{x\to 0} \frac{\tan x - x}{\sin x - x} = \boxed{\text{use the L'Hopital's rule}} =$$

$$\lim_{x\to 0} \frac{\sec^2 x - 1}{\cos x - 1} = \boxed{\text{use the L'Hopital's rule}} =$$

$$\lim_{x\to 0} \frac{2\sec^2 x \tan x}{-\sin x} = 2 \cdot \lim_{x\to 0} \frac{\sin x}{-\sin x \cos^3 x} = 2 \cdot \lim_{x\to 0} \frac{1}{-\cos^3 x} = -2.$$

Problem 4. Compute the integral

$$\int \frac{x^3 - 8x^2 + 21x - 19}{x^2 - 6x + 9} \, \mathtt{d}x.$$

Space for your solution:

1. Divide the polynomial $x^3 - 8x^2 + 21x - 19$ by the polynomial $x^2 - 6x + 9$.

$$\begin{array}{r|l} & x - 2 \\ \hline x^2 - 6x + 9 & x^3 - 8x^2 + 21x - 19 \\ & x^3 - 6x^2 + 9x \\ \hline & -2x^2 + 12x - 19 \\ & -2x^2 + 12x - 18 \\ \hline & -1 \end{array}$$

   The above long division means that

$$\int \frac{x^3 - 8x^2 + 21x - 19}{x^2 - 6x + 9} \, \mathtt{d}x = \int \left(x - 2 + \frac{-1}{x^2 - 6x + 9} \right) \, \mathtt{d}x.$$

2. Integrate $x - 2$:

$$\int (x - 2) \, \mathtt{d}x = \frac{(x - 2)^2}{2} + C.$$

3. Using the method of partial fractions (page 21 of the notes), integrate $\frac{-1}{x^2 - 6x + 9}$:

$$\frac{-1}{x^2 - 6x + 9} = \frac{-1}{(x - 3)^2} = \frac{A}{x - 3} + \frac{B}{(x - 3)^2}.$$

   It is clear that our fraction itself happens to be a partial fraction (in other words, we can take $A = 0$, $B = -1$.

$$\int \frac{-1}{(x - 3)^2} \, \mathtt{d}x = -\int (x - 3)^{-2} \, \mathtt{d}(x - 3) = \frac{1}{x - 3} + C.$$

4. Add the two integrals from above:

$$\int \frac{x^3 - 8x^2 + 21x - 19}{x^2 - 6x + 9} \, \mathtt{d}x = \frac{(x - 2)^2}{2} + \frac{1}{x - 3} + C.$$

Problem 5. Consider the function $f(x) = \sqrt[3]{x}$.

(1). Which point (or points) from the domain of the function f would be a good choice for the center of a Taylor polynomial for f and why?

Space for your solution:

Consider the successive-order derivatives of $f(x)$ up to the order 5 (this last number is chosen to suit the next sub-problem):

$$\left(\frac{\mathtt{d}}{\mathtt{d}x}\right)^0 f(x) = f(x) = \sqrt[3]{x} = x^{\frac{1}{3}},$$

$$\left(\frac{\mathtt{d}}{\mathtt{d}x}\right)^1 f(x) = \left(x^{\frac{1}{3}}\right)' = \frac{1}{3}\, x^{\frac{1}{3}-1} = \frac{1}{3}\, x^{-\frac{2}{3}},$$

$$\left(\frac{\mathtt{d}}{\mathtt{d}x}\right)^2 f(x) = \left(\frac{1}{3}\, x^{-\frac{2}{3}}\right)' = \frac{1}{3} \cdot \left(-\frac{2}{3}\right)\, x^{-\frac{2}{3}-1} = -\frac{2}{9}\, x^{-\frac{5}{3}},$$

$$\left(\frac{\mathtt{d}}{\mathtt{d}x}\right)^3 f(x) = \left(-\frac{2}{9}\, x^{-\frac{5}{3}}\right)' = -\frac{2}{9} \cdot \left(-\frac{5}{3}\right)\, x^{-\frac{5}{3}-1} = \frac{10}{27}\, x^{-\frac{8}{3}},$$

$$\left(\frac{\mathtt{d}}{\mathtt{d}x}\right)^4 f(x) = \left(\frac{10}{27}\, x^{-\frac{8}{3}}\right)' = \frac{10}{27} \cdot \left(-\frac{8}{3}\right)\, x^{-\frac{8}{3}-1} = -\frac{80}{81}\, x^{-\frac{11}{3}},$$

$$\left(\frac{\mathtt{d}}{\mathtt{d}x}\right)^5 f(x) = \left(-\frac{80}{81}\, x^{-\frac{11}{3}}\right)' = -\frac{80}{81} \cdot \left(-\frac{11}{3}\right)\, x^{-\frac{11}{3}-1} = \frac{880}{243}\, x^{-\frac{14}{3}} \ldots$$

From these derivatives we can see the general pattern that the higher order derivative of $f(x)$ will always be a negative fractional power of x. We are looking for an internal point in the domain of these derivatives that would allow an easy evaluation of those derivatives. Since fractional powers rule out negative bases, zero falls on the boundary of these domains and must be ruled out (in spite of being a good input for computing these higher order derivatives). Another good candidate is $x = 1$ where all the powers of x turn into 1, making the rest of the computation easy:

$\left(\frac{\mathtt{d}}{\mathtt{d}x}\right)^0 f(1) = 1$, $\left(\frac{\mathtt{d}}{\mathtt{d}x}\right)^1 f(1) = \frac{1}{3}$, $\left(\frac{\mathtt{d}}{\mathtt{d}x}\right)^2 f(1) = -\frac{2}{9}$, $\left(\frac{\mathtt{d}}{\mathtt{d}x}\right)^3 f(x) = \frac{10}{27}$, $\left(\frac{\mathtt{d}}{\mathtt{d}x}\right)^4 f(x) = -\frac{80}{81}$, $\left(\frac{\mathtt{d}}{\mathtt{d}x}\right)^5 f(x) = \frac{880}{243} \ldots$

Thus, the best candidate for the center of the Taylor polynomial is $x_0 = 1$.

(2). Compute the Taylor polynomial of degree 5 for the function f at one of the points you found in part (1).

Space for your solution:

Applying the general formula for the Taylor polynomial

$$\sum_{i=0}^{n} \left(\frac{\mathtt{d}}{\mathtt{d}x}\right)^{i} f(x_0) \cdot \frac{(x-x_0)^i}{i!}$$

to the case of our specific $f(x)$ and x_0, we get:

$$1 + \frac{x-1}{3} - \frac{2(x-1)^2}{9 \cdot 2!} + \frac{10(x-1)^3}{27 \cdot 3!} - \frac{80(x-1)^4}{81 \cdot 4!} + \frac{880(x-1)^5}{880 \cdot 5!}.$$

(3). Determine the Taylor series of the function f.

Space for your solution:

We need to find the general pattern for higher-order derivatives of $f(x)$ at $x = 1$. It is easy to see that the i-th coefficient can be computed as the product

$$\prod_{j=0}^{i-1} \left(\frac{1}{3} - j\right)$$

This product, called the *falling factorial* of $\frac{1}{3}$, is denoted $\left(\frac{1}{3}\right)_i$. Using this notation, we can write the Taylor series of $f(x) = \sqrt[3]{x}$ at $x = 1$ as:

$$\sum_{i=0}^{+\infty} \left(\frac{1}{3}\right)_i \cdot \frac{(x-1)^i}{i!}$$

(4). Find the radius of convergence of the series you found in part (3).

Space for your solution:

Denote $a_i = \frac{\left(\frac{1}{3}\right)_i}{i!}$ the i-th coefficient of the above Taylor series. Applying the ratio test to this Taylor series, we get:

$$\lim_{i \to +\infty} \left| \frac{a_i}{a_{i-1}} \right| = \lim_{i \to +\infty} \left| \frac{4}{3i} - 1 \right| \cdot |x| = |x|,$$

resulting in the radius of convergence $R = 1$.

(5). Give an estimate of the error term of the Taylor polynomial of the function f within the interval of convergence. The estimate must be computable by arithmetic operations only and must go to zero as the degree of the Taylor polynomial goes to infinity.

Space for your solution:

The exact error of the Taylor polynomial of degree n

$$(-1)^n \cdot \frac{1}{n!} \int_{x_0}^{x} (t-x)^n \cdot \left(\frac{\mathtt{d}}{\mathtt{d}x}\right)^{n+1} f(t)\ dt$$

turns into

$$(-1)^n \cdot \frac{1}{n!} \int_{1}^{x} (t-x)^n \cdot \left(\frac{1}{3}\right)_{n+1} \cdot x^{\frac{1}{3}-n-1}\ dt = (-1)^n \cdot \int_{1}^{x} (t-x)^n \cdot \frac{1}{3} \cdot \prod_{j=1}^{n} \left(\frac{\frac{1}{3}-j}{j}\right) \cdot x^{\frac{1}{3}-n-1}\ dt.$$

Since the absolute value of this integral is no bigger than that of the

$$\int_{t=1}^{x} (t-x)^n\ dt = \int_{z=1-x}^{0} z^n\ dz = \left.\frac{z^{n+1}}{n+1}\right|_{z=1-x}^{0} = -\frac{(1-x)^{n+1}}{n+1},$$

which in turn is no bigger than $\frac{1}{n+1}$, we can use the last number as the estimate of the error term for the Taylor polynomial we found.

(6). Propose a method for finding the values of f at the points outside of the interval of convergence.

Space for your solution:

For any $x \in (0, 2)$ we can use the Taylor polynomial approximation directly. So, our task in this sub-problem is to find a computation method for any $x \in [2, +\infty)$. Given any $x \geq 2$, take an exact cube c^3 that is bigger than x, so that the fraction $\frac{x}{c^3} < 1$. Then we can use the Taylor polynomial for $\frac{x}{c^3}$ that would give us the approximate value of

$$\sqrt[3]{\frac{x}{c^3}} = \frac{\sqrt[3]{x}}{c}.$$

Multiplying it by c, we get the needed approximation of $\sqrt[3]{x}$.